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Quantum Energy Lines
and the optimal output
ergotropy problem

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Quantum Energy Lines and the optimal output ergotropy problem

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We study the transferring of useful energy (work) along a transmission line that allows for partial preservation of quantum coherence. As a figure of merit we adopt the maximum values that ergotropy, total ergotropy, and non-equilibrium free-energy attain at the output of the line for an assigned input energy threshold. For Phase-Invariant Bosonic Gaussian Channels (BGCs) models, we show that coherent inputs are optimal. For (one-mode) not Phase-Invariant BGCs we solve the optimization problem under the extra restriction of Gaussian input signals.
Quantum Technologies

Quantum Thermodynamics

Equilibration/Thermalization Processes

Quantum Engines
(Energy production)

Quantum Batteries
(Energy storage)

Goold et al. JMP 49 (2016)

Can we use “quantum signals” to transfer energy?

What are the suitable (realistic) models?

Are they efficient?

What are the proper figure of merit to address the problem?
Our Results (in brief)

❖ A CV variable model for QEL (e.m. transmission lines)

❖ Show that COHERENT states of the radiation are optimal inputs: they allow one to transfer the maximum amount of “useful energy”
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❖ A CV variable model for QEL (e.m. transmission lines)

❖ Show that COHERENT states of the radiation are optimal inputs: they allow one to transfer the maximum amount of “useful” energy
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QUANTUM ENERGY LINES
QUANTUM ENERGY LINES

“Any physical media that allows one to transfer energy from one location A to a second location B while maintaining a certain degree of quantum coherence in the transmitted signals.”

Optical fibers
Wave guides
Free-space e.m. communication
Nanowires
Metallic leads
...

Figure 1. Transferring classical data \( x \in X \) via a quantum information carrier. The first stage of the process requires the encoding of \( x \) into a quantum state \( \rho_x \) of the carrier (c-q mapping); then the carrier propagates along the communication line and its state gets transformed into the density matrix \( \rho'_x \) (q-q mapping); finally there is the decoding stage where the receiver of the message tries to recover \( x \) by performing some measurement on \( \rho'_x \) and obtaining the classical outcome \( y \) (q-c mapping).

To preserve the statistical structure of the process, a mixed input state of \( X \) defined by the probability distribution \( P = \{ p_x \} \) is mapped by the c-q channel into the density operator \( \rho = \sum_x p_x \rho_x \).

On the other hand, the decoding stage of Fig. 1 is characterized by a quantum-classical (q-c) mapping that establishes the probabilities of the output letters \( y \in Y \) corresponding to the quantum state of the carrier emerging from the quantum communication line. Such a mapping is implemented by a quantum measurement and it is characterized by assigning the probability distribution \( p_y(\rho) \) that, given a generic state \( \rho \) of the carrier, defines the statistics of the possible measurement outcomes. It can be shown \([87]\) that the linear dependence of \( p_y(\rho) \) resulting from the preservation of mixtures, along with general properties of probabilities, implies the following functional structure \( p_y(\rho) = \text{Tr} \rho M_y \), \( (6) \) where \( \{ M_y \} \) is a collection of Hermitian operators in \( H \) with the properties:

\( M_y \geq 0 \), \( \sum_y M_y = I \).

Any such collection is called probability operator-valued measure (POVM), or, in the modern quantum phenomenology, just observable \([88]\). Later we shall explain how the operators \( M_y \) arise from the dynamical description of a measurement process. An observable is called sharp if \( M_y \) are mutually orthogonal projection operators, i.e. \( M_y^2 = M_y \), \( M_y M_y' = 0 \) if \( y \neq y' \). In this case (6) amounts to the well-known “Born-von Neumann statistical postulate” \([42, 181]\).
Quantum Channel formalism (CPT formalism) \[\Rightarrow \quad \text{QUANTUM COMMUNICATION}\]

\[
\rho_x \rightarrow \Phi \rightarrow \rho'_x
\]

\[\Phi = \text{transformation which maps the (quantum) state of the input signals into the associated (quantum) states at the output of the communication line}\]

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TRACE PRESERVING COMPLETELY POSITIVE LINEAR SUPEROPERATORS

Generalization of the notion of SCATTERING matrices, to the case where signals undergo to NOISE (fluctuations, thermalization effects, spurious collisions, losses, etc.)
Physical Model of the QEL

Input signal
\( (\hat{H} \geq 0) \)

\( \hat{\rho} \)

Location A

\( \Phi \)

\( \hat{\rho}' = \Phi[\hat{\rho}] \)

Location B

Output signal

\[ E_{in} = \text{Tr}[\hat{\rho}\hat{H}] \quad \rightarrow \quad E_{out} = \text{Tr}[\hat{\rho}'\hat{H}] = \text{Tr}[\Phi[\hat{\rho}]\hat{H}] \]

Which input signals with assigned input energy \( E \) produce output signals that ensure the highest value of “useful energy”?
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“USEFUL” ENERGY
"USEFUL" ENERGY = EXTRACTABLE WORK

\[ \Delta E = Q - W \]

- Variation of the Internal energy of a system
- Heat added to the system
- Work done by the system
**“USEFUL” ENERGY = EXTRACTABLE WORK**

\[
\Delta E = E_{out} - \epsilon_0 = Q - W
\]

(ground state energy of the energy carrier)
"USEFUL" ENERGY = EXTRACTABLE WORK in a QUANTUM SETTING

\[ \Delta E = E(\hat{\rho}') - E(\hat{\rho}) = Q - W \]

The answer depends on the type of allowed transformation and on the resources we invest in the process.

WE FOCUS ON THREE DIFFERENT SCENARIOS
1) If you allow only MODULATIONS of the system Hamiltonian (no thermal contact with external baths),

\[
E(\hat{\rho}) := \text{Tr}[\hat{\rho}\hat{H}] - \min_V \text{Tr}[V\hat{\rho}V^\dagger \hat{H}]
\]

EW = ERGOTROPY

Lenard JSP 19 (1978)
Petz & Woronowicz CMP (1978)
Alicki JPA (1979)
Allahverdjan et al. EPL 67 (2004)
ERGOTROPY

NON LINEAR functional of

\[ \hat{\rho} = \sum_{j=0} \lambda_j |j\rangle\langle j| \quad \text{and} \quad \hat{H} = \sum_{j=0} \epsilon_j |\epsilon_j\rangle\langle\epsilon_j| \]

\[ \mathcal{E}(\hat{\rho}) = E(\hat{\rho}) - \sum_{j=0} \lambda_j^{(\uparrow)} \epsilon_j \]

\[ 0 \leq \mathcal{E}(\hat{\rho}) \leq E(\hat{\rho}) = \text{Tr}[\hat{\rho}\hat{H}] \]

\[ \mathcal{E}(\hat{\rho}) = 0 \quad \text{for PASSIVE states (i.e. diagonal in the energy eigen-basis with decreasing spectrum)} \]

\[ \mathcal{E}(\hat{\rho}) = E(\hat{\rho}) \quad \text{if and only if the state is PURE} \]
2) If you allow only MODULATIONS of the system Hamiltonian of $N$ copies of the system (no thermal contact with external baths),

$$\text{EW per copies} = \text{N-ERGOTROPY}$$

$$\mathcal{E}^{(N)}(\hat{\rho}) = \frac{\mathcal{E}(\hat{\rho} \otimes^N)}{N} \geq \mathcal{E}(\hat{\rho})$$

Lenard JSP 19 (1978)
Petz & Woronowicz CMP (1978)
Alicki JPA (1979)
Allahverdjan et al. EPL 67 (2004)
**TOTAL ERGOTROPY**

\[
\mathcal{E}_\text{tot}(\hat{\rho}) = \lim_{N \to \infty} \mathcal{E}^{(N)}(\hat{\rho}) = E(\hat{\rho}) - E(\hat{\omega}_\beta)
\]

Gibbs Thermal state of the system with THE SAME entropy of \( \hat{\rho} \)

\[
\hat{\omega}_\beta = \frac{e^{-\beta \hat{H}}}{Z(\beta)}
\]

\[
S(\hat{\omega}_\beta) = S(\hat{\rho})
\]
3) If you allow the agent to put the system in thermal contact with an external source at temperature $T = 1/\beta$

\[
\mathcal{F}_\beta(\hat{\rho}) = E(\hat{\rho}) - \frac{1}{\beta} S(\hat{\rho})
\]

Brandao et al, PNAS 112 (2015)

N.B. for $T=0$ \( \mathcal{F}_\beta(\hat{\rho}) \rightarrow E(\hat{\rho}) \)
THE PROBLEM

FIND which are the input states with fixed input energy that yield the maximum value of the EW at the output of the transmission line

\[ EW_E(\Phi) = \max_{\hat{\rho} \in \mathcal{H}_E} EW(\Phi(\hat{\rho})) =? \]

\[ \mathcal{H}_E = \{ \hat{\rho} : E(\hat{\rho}) \leq E \} \]

EW = ERGOTROPY

EW = TOTAL ERGOTROPY

EW = NON EQUILIBRIUM FREE ENERGY
3

RESULTS
collection of $n$ e.m. modes

\[ \hat{a}_i, \hat{a}^\dagger_i = \text{annihilation and creation operators of the i-th mode} \]
\[ (i = 1, \cdots, n) \]
\[ [\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij} \]
\[ [\hat{a}_i, \hat{a}_j] = 0 \]

\[ \hat{H} = \sum_{i=1}^{n} \hbar \omega_i \hat{a}^\dagger_i \hat{a}_i \]
\[ \omega_i = \text{frequency of the i-th mode} \]
\[ (\omega_i \approx \omega_0) \]
collection of $n$ e.m. modes

Location A  

NOISE

Location B

ATTENUATION (LOSSES)

THERMALIZATION

AMPLIFICATION (GAIN)

“PHASE-INSENSITIVE GAUSSIAN BOSONIC CHANNELS”

Holevo and Giovannetti RMP (2012)
Given \( \Phi \) a PI-BGC, for any \( E \) the maximum output Extractable Work is achieved by a (multimode) coherent state

\[
EW_E(\Phi) = \max_{\hat{\rho} \in \mathcal{H}_E} EW(\Phi(\hat{\rho}))
\]

is achieved by a (multimode) coherent state

(multimode) COHERENT STATES

\[
\hat{\rho}_{coh} = |\Psi\rangle \langle \Psi|
\]

\[
|\Psi\rangle = \hat{D}(\vec{m})|\emptyset\rangle \quad \text{DISPLACED VACUUM STATES}
\]
Sketch of the proof:

0) Notice that all our 3 examples of EW can be expressed as

\[ \text{EW}(\Phi(\rho)) = E(\Phi(\rho)) - (\text{something} \geq 0) \]

with the \text{something} terms being all Shur concave functionals of the input state.

1) show that for any input state \( \hat{\rho} \) there exists a properly chosen coherent input \( \hat{\rho}_{\text{cohe}} \) with the same initial energy which at the output of the channel provides a signal with mean energy that is larger than or equal to the output mean energy of \( \hat{\rho} \)

\[ E(\hat{\rho}) = E(\hat{\rho}_{\text{cohe}}) \quad E(\Phi(\hat{\rho})) \leq E(\Phi(\hat{\rho}_{\text{cohe}})) \]

2) invoke the fact that at the output of a PI-GBC coherent inputs MAJORIZE (are more ordered than) all other signals

Examples

\[ EW_E(\Phi) = \eta E \quad \text{Single mode} \]

\[ \eta \in [0, 1] \quad \text{transmissivity} \]

\[ EW_E(\Phi) = \kappa E \quad \text{Single mode} \]

\[ \kappa \geq 1 \quad \text{gain} \]
GAUSSIAN BOSONIC CHANNELS
Which are NOT PHASE-INSSENSITIVE

Squeezed thermal environment

Location A  Location B

Optimal inputs are not known in this case

Coherent input states ARE certainly NOT optimal

Properly squeeed inputs for instance perform better than coherent states
CONCLUSIONS

- QEL models have being introduced
- Optimization problem has been presented based on the notion of EW
- For CV systems (realistic noise model) optimal solution have been presented

FUTURE DIRECTIONS

- Extended the analysis to other QEL models
- Introduce different figure of merit (power, instead of energy, work distribution)
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Extras
collection of $n$ e.m. modes

Location A \[\Phi\] Location B

**PHASE-INSSENSITIVE GAUSSIAN BOSONIC CHANNELS**

(loss, thermalization, amplification noise, etc)

\[
\hat{r} = (\hat{q}_1, \cdots, \hat{q}_n; \hat{p}_1, \cdots, \hat{p}_n)^T
\]

vector of the canonical coordinates

\[
\hat{D}(\vec{x}) = \exp[\hat{r} \cdot \vec{x}]
\]

displacement operator

\[
\hat{D}(\vec{x}) = \exp[\hat{r} \cdot \vec{x}]
\]

\[
\chi(\hat{\rho}; \vec{x}) = \text{Tr}[\hat{\rho} \hat{D}(\vec{x})]
\]

**CHARACTERISTIC FUNCTION**

(Fourier transform of the Wigner distribution of the state)
\[ \chi(\hat{\rho}; \vec{x}) \]

GAUSSIAN BOSONIC CHANNELS

\[ \chi(\Phi(\hat{\rho}); \vec{x}) = \chi(\hat{\rho}; X^T \vec{x}) \exp\left[-\frac{1}{4} \vec{x}^T Y \vec{x} + i \vec{v}^T \vec{x}\right] \]

\[ \vec{v} \in R^{2n} \]

\[ X, Y \in R^{2n \times 2n} \quad Y \geq i(\gamma - X\gamma X^\dagger) \quad \text{(symplectic matrix)} \]

\[ \Phi(e^{-i\hat{H}t} \hat{\rho} e^{i\hat{H}t}) = e^{-i\hat{H}t} \Phi(\hat{\rho}) e^{i\hat{H}t} \quad \text{PHASE-INSENSITIVE Condition} \]
Bosonic Gaussian Channels (BGC)

“Completely Positive Trace Preserving (CPTP) Super-operators $\Phi$ which map Gaussian states into Gaussian states”

$$\hat{\rho} \rightarrow \Phi(\hat{\rho})$$

Gaussian states = Gibbs states of arbitrary Hamiltonians which are quadratic in the field operators

$$[\hat{q}, \hat{p}] = i \quad \hat{r} := (\hat{q}, \hat{p})^T$$

Covariance matrix

$$\phi_G(\hat{\rho}; z) = \exp[-\frac{1}{4} z^T \sigma z + im^T \cdot z]$$

symplectic form

characteristic function of the state

$$\phi(\hat{\rho}; z) := \text{Tr} \left[ \hat{\rho} e^{i\hat{r} \cdot z} \right]$$

Weil (displacement) operator

$$z \in \mathbb{R}^2$$

Holevo, Werner PRA 2001